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Dynamics of parametric processes with a trilinear hamiltonian

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Abstract. The quantum dynamics of a parametric interaction of the electromagnetic field with a nonlinear medium is considered. The three nonlinear coupled Heisenberg equations of motion are solved under the short-time approximation. The characteristic function for the normal ordering rule of association is evaluated. This function is then used to obtain the time dependence of the density operator. Assuming that the initial state of the system is a coherent state, explicit expressions for the diagonal coherent state representation of the reduced density operators for the pump as well as for the signal mode are obtained. For the pump mode, the initially coherent state remains coherent, whereas for the signal mode the diagonal coherent state representation is found to be a gaussian distribution whose variance is proportional to the average number of photons initially present in the pump mode.

1. Introduction

Oscillations of a system with time-varying parameters have been known for a long time (Rayleigh 1883) and are commonly called parametric oscillations. These oscillations have played a central role in several physical phenomena of interest which include frequency conversion in nonlinear media, coherent Raman and Brillouin scattering, spontaneous and stimulated emissions of radiation, super-radiance, etc. A simple quantum-mechanical model for parametric oscillations was first suggested by Louisell *et al* (1961). Based on this model a detailed study of the quantum statistics has been carried out (Mollow and Glauber 1967, Graham and Haken 1968, Tucker and Walls 1969a, b) under the parametric approximation. This approximation consists of assuming the pump mode to be quite intense so that the average number of photons in this mode can be considered as constant. However, it is observed in this case that the average number of photons in the signal as well as in the idler mode grows exponentially with time. For this reason, the model is not very satisfactory. Recently Tucker and Walls (1969a,b) have suggested a parametric interaction where the interaction hamiltonian is trilinear and reduces under the parametric approximation to the bilinear hamiltonian as suggested by Louisell *et al* (1961). The Heisenberg equations of motion obtained using this hamiltonian are coupled nonlinear equations, and have not been solved exactly. However, using the fact that the free and the interaction parts of the hamiltonian commute with each other, Walls and Barakat (1970) carried out simultaneous diagonalization of the two parts numerically. They found that the number of photons in the signal or the idler mode follows an almost periodic variation.

In the present investigation we solve the three nonlinear coupled equations of motion under the short-time approximation. Under this approximation, we factor out the harmonic time variation of the annihilation operator of each mode and expand the remaining slowly-varying part in a Taylor series retaining terms only up to those quadratic in time.

The calculations are thus valid if the time for which the interaction among different modes is on is sufficiently small. The solutions are then used to obtain the average and the variance of the number of photons in different modes. The time evolution of the reduced density operators for different modes is also considered. In particular we find that if the initial state of the system is a coherent state, the reduced diagonal coherent state representation of the signal mode is a gaussian distribution whereas the reduced state of the pump mode remains coherent. The calculations have not been carried out separately for the idler mode. This is because the hamiltonian is completely symmetric in the signal and the idler modes, and therefore the results for the idler mode are obtained by simply interchanging the idler and the signal mode parameters. We also find the diagonal coherent state representation for the combined signal and the idler modes. It is found that the two modes behave independently.

2. The trilinear model hamiltonian

We consider the model hamiltonian of Tucker and Walls (1969a,b). Assuming that there are only three modes interacting with each other, we write the hamiltonian in the form†

$$\hat{H} = \hat{H}_0 + \hat{H}_1, \tag{2.1}$$

where

$$\hat{H}_0 = \hbar[\omega_a \hat{a}^\dagger(t)\hat{a}(t) + \omega_b \hat{b}^\dagger(t)\hat{b}(t) + \omega_c \hat{c}^\dagger(t)\hat{c}(t)] \tag{2.2}$$

and

$$\hat{H}_1 = \hbar k[\hat{a}^\dagger(t)\hat{b}(t)\hat{c}(t) + \hat{a}(t)\hat{b}^\dagger(t)\hat{c}^\dagger(t)]. \tag{2.3}$$

The coupling constant k is taken to be real and the energy conserving condition

$$\omega_a = \omega_b + \omega_c \tag{2.4}$$

is assumed to hold. The three modes are labelled by the subscripts a, b and c. The annihilation and creation operators $\hat{a}, \hat{a}^\dagger, \hat{b}, \hat{b}^\dagger$ and $\hat{c}, \hat{c}^\dagger$ satisfy the commutation relations

$$[\hat{a}, \hat{a}^\dagger] = [\hat{b}, \hat{b}^\dagger] = [\hat{c}, \hat{c}^\dagger] = 1, \tag{2.5}$$

$$[\hat{a}, \hat{b}] = [\hat{a}, \hat{c}] = [\hat{b}, \hat{c}] = [\hat{a}, \hat{b}^\dagger] = [\hat{b}, \hat{c}^\dagger] = [\hat{a}, \hat{c}^\dagger] = 0. \tag{2.6}$$

The hamiltonian (2.1) describes the parametric amplification if one identifies a, b and c as the pump mode, the signal mode and the idler mode respectively. Various other phenomena of physical interest can be described by the same model hamiltonian by suitably identifying these modes. Thus, for example, if the signal and the idler mode frequencies are identical ($\omega_b = \omega_c$) the hamiltonian describes second harmonic generation (Walls and Tindle 1972). If, on the other hand, we identify 'a' as the idler mode, 'b' as the signal mode and 'c' as the pump mode, the hamiltonian (2.1) will describe frequency conversion (Tucker and Walls 1969a, b).

The hamiltonian describing processes such as spontaneous emission, stimulated emission, super-radiance, etc, in which we study the problem of emission of radiation

† For the sake of clarity, all operators are indicated by a circumflex.

from a system of two-level atoms, could also be transformed in the form (2.1) by representing angular momentum operators in terms of boson operators (Schwinger 1965). The two hamiltonians become formally identical by taking the pump mode as the state corresponding to the upper atomic level, the signal mode as the state corresponding to the lower atomic level and the idler mode corresponding to the emitted photon (Bonifacio and Preparata 1970).

3. Equations of motion

We consider now the dynamics of the system described by the model hamiltonian (2.1); namely

$$\hat{H} = \hbar[\omega_a \hat{a}^\dagger(t)\hat{a}(t) + \omega_b \hat{b}^\dagger(t)\hat{b}(t) + \omega_c \hat{c}^\dagger(t)\hat{c}(t)] + \hbar k[\hat{a}^\dagger(t)\hat{b}(t)\hat{c}(t) + \hat{a}(t)\hat{b}^\dagger(t)\hat{c}^\dagger(t)]. \tag{3.1}$$

The Heisenberg equation of motion for any operator $\hat{G}$, which does not depend on time explicitly, is given by

$$i\hbar \frac{d\hat{G}}{dt} = [\hat{G}, \hat{H}]. \tag{3.2}$$

Using equations (3.2) and (3.1), we obtain the following nonlinear coupled equations for the annihilation operators $\hat{a}(t)$, $\hat{b}(t)$ and $\hat{c}(t)$:

$$i\frac{d\hat{a}(t)}{dt} = \omega_a \hat{a}(t) + k\hat{b}(t)\hat{c}(t), \tag{3.3}$$

$$i\frac{d\hat{b}(t)}{dt} = \omega_b \hat{b}(t) + k\hat{a}(t)\hat{c}^\dagger(t), \tag{3.4}$$

$$i\frac{d\hat{c}(t)}{dt} = \omega_c \hat{c}(t) + k\hat{a}(t)\hat{b}^\dagger(t). \tag{3.5}$$

If we denote the number operators $\hat{a}^\dagger(t)\hat{a}(t)$, $\hat{b}^\dagger(t)\hat{b}(t)$ and $\hat{c}^\dagger(t)\hat{c}(t)$ by $\hat{N}_a(t)$, $\hat{N}_b(t)$ and $\hat{N}_c(t)$ respectively, we may readily verify that

$$\frac{d}{dt}[\hat{N}_a(t) + \hat{N}_b(t)] = 0, \tag{3.6}$$

$$\frac{d}{dt}[\hat{N}_a(t) + \hat{N}_c(t)] = 0, \tag{3.7}$$

$$\frac{d}{dt}[\hat{N}_b(t) - \hat{N}_c(t)] = 0. \tag{3.8}$$

Thus we find that $\hat{N}_a(t) + \hat{N}_b(t)$, $\hat{N}_a(t) + \hat{N}_c(t)$ and $\hat{N}_b(t) - \hat{N}_c(t)$ are constants of the motion. These Manley–Rowe relations (Louisell 1960) are simultaneously satisfied if we require that for every photon that is annihilated from mode A, there is created one photon each in mode B and C.

Let us introduce the slowly-varying operators $\hat{A}(t)$, $\hat{B}(t)$ and $\hat{C}(t)$ defined by the relations

$$\hat{a}(t) = \hat{A}(t)\exp(-i\omega_a t), \tag{3.9}$$

$$\hat{b}(t) = \hat{B}(t)\exp(-i\omega_b t), \tag{3.10}$$

$$\hat{c}(t) = \hat{C}(t)\exp(-i\omega_c t). \tag{3.11}$$

From equations (3.3)–(3.5) and (3.9)–(3.11), we find that the operators $\hat{A}(t)$, $\hat{B}(t)$ and $\hat{C}(t)$ satisfy the following equations:

$$i\frac{d\hat{A}(t)}{dt} = k\hat{B}(t)\hat{C}(t), \tag{3.12}$$

$$i\frac{d\hat{B}(t)}{dt} = k\hat{A}(t)\hat{C}^\dagger(t), \tag{3.13}$$

$$i\frac{d\hat{C}(t)}{dt} = k\hat{A}(t)\hat{B}^\dagger(t). \tag{3.14}$$

As remarked earlier, equations (3.12)–(3.14) may be used to describe parametric as well as other radiation processes by suitably identifying the various modes. In what follows, we are considering the parametric amplification by identifying A as the pump mode, B as the signal mode and C as the idler mode.

4. The short-time approximation

Let us assume that the time period for which the interaction is present is sufficiently small, so that we may expand each of the annihilation operators $\hat{A}(t)$, $\hat{B}(t)$ and $\hat{C}(t)$ in a Taylor series and retain terms only up to those quadratic in time. This is justified since the rapid exponential time variation of the operators $\hat{a}(t)$, $\hat{b}(t)$ and $\hat{c}(t)$ is already factored out in equations (3.9)–(3.11). We therefore write

$$\hat{A}(t) = \hat{A}_0 + t\hat{A}'_0 + \tfrac{1}{2}t^2\hat{A}''_0, \tag{4.1}$$

$$\hat{B}(t) = \hat{B}_0 + t\hat{B}'_0 + \tfrac{1}{2}t^2\hat{B}''_0, \tag{4.2}$$

$$\hat{C}(t) = \hat{C}_0 + t\hat{C}'_0 + \tfrac{1}{2}t^2\hat{C}''_0, \tag{4.3}$$

where the subscript 0 indicates the values at the initial time $t = 0$. Substituting these equations in (3.12)–(3.14) and equating coefficients of the various powers of t, we obtain the following relations:

$$\hat{A}(t) = \hat{A}_0 - ikt\hat{B}_0\hat{C}_0 - \tfrac{1}{2}k^2t^2\hat{A}_0(\hat{B}_0\hat{B}_0^\dagger + \hat{C}_0^\dagger\hat{C}_0), \tag{4.4}$$

$$\hat{B}(t) = \hat{B}_0 - ikt\hat{A}_0\hat{C}_0^\dagger + \tfrac{1}{2}k^2t^2\hat{B}_0(\hat{A}_0^\dagger\hat{A}_0 - \hat{C}_0^\dagger\hat{C}_0), \tag{4.5}$$

$$\hat{C}(t) = \hat{C}_0 - ikt\hat{A}_0\hat{B}_0^\dagger + \tfrac{1}{2}k^2t^2\hat{C}_0(\hat{A}_0^\dagger\hat{A}_0 - \hat{B}_0^\dagger\hat{B}_0). \tag{4.6}$$

Relations (4.4)–(4.6) may be used to evaluate the time dependence of the average number of photons in different modes. Since $\hat{N}_a(t) = \hat{a}^\dagger(t)\hat{a}(t) = \hat{A}^\dagger(t)\hat{A}(t)$, we find that

$$\hat{N}_a(t) = \hat{N}_{a0} - k^2t^2\hat{N}_{a0}(\hat{N}_{b0} + \hat{N}_{c0} + 1) + k^2t^2\hat{N}_{b0}\hat{N}_{c0} - ikt(\hat{A}_0^\dagger\hat{B}_0\hat{C}_0 - \hat{A}_0\hat{B}_0^\dagger\hat{C}_0^\dagger), \tag{4.7}$$

$$\hat{N}_b(t) = \hat{N}_{b0} + k^2t^2\hat{N}_{a0}(\hat{N}_{b0} + \hat{N}_{c0} + 1) - k^2t^2\hat{N}_{b0}\hat{N}_{c0} + ikt(\hat{A}_0^\dagger\hat{B}_0\hat{C}_0 - \hat{A}_0\hat{B}_0^\dagger\hat{C}_0^\dagger), \tag{4.8}$$

$$\hat{N}_c(t) = \hat{N}_{c0} + k^2t^2\hat{N}_{a0}(\hat{N}_{b0} + \hat{N}_{c0} + 1) - k^2t^2\hat{N}_{b0}\hat{N}_{c0} + ikt(\hat{A}_0^\dagger\hat{B}_0\hat{C}_0 - \hat{A}_0\hat{B}_0^\dagger\hat{C}_0^\dagger). \tag{4.9}$$

The Manley–Rowe relations (3.6)–(3.8) are of course obviously satisfied. Let n_a, n_b and n_c be the average number of photons present initially in different modes. The average number of photons and the variance at any later time are then given by

$$\bar{n}_\mu(t) = \langle n_a, n_b, n_c|\hat{N}_\mu(t)|n_a, n_b, n_c\rangle, \tag{4.10}$$

$$\langle \Delta n_\mu^2\rangle = \langle n_a, n_b, n_c|\hat{N}_\mu^2(t)|n_a, n_b, n_c\rangle - [\bar{n}_\mu(t)]^2; \qquad \mu = a, b, c. \tag{4.11}$$

We thus find on using (4.7)–(4.9) that

$$\bar{n}_a(t) = n_a - k^2t^2n_a(n_b+n_c+1)+k^2t^2n_bn_c, \tag{4.12}$$

$$\bar{n}_b(t) = n_b + k^2t^2n_a(n_b+n_c+1)-k^2t^2n_bn_c, \tag{4.13}$$

$$\bar{n}_c(t) = n_c + k^2t^2n_a(n_b+n_c+1)-k^2t^2n_bn_c \tag{4.14}$$

and

$$\langle \Delta n_a^2\rangle = \langle \Delta n_b^2\rangle = \langle \Delta n_c^2\rangle$$
$$= [2n_an_bn_c + n_a(n_b+n_c+1)+n_bn_c]k^2t^2. \tag{4.15}$$

One may further verify that

$$\langle \Delta n_b\, \Delta n_c\rangle = -\langle \Delta n_a\, \Delta n_c\rangle = -\langle \Delta n_a\, \Delta n_b\rangle = \langle \Delta n_a^2\rangle. \tag{4.16}$$

If initially there are no photons present in the idler mode ($n_c = 0$), we find that

$$\bar{n}_a(t) = n_a - k^2t^2n_a(n_b+1), \tag{4.17}$$

$$\bar{n}_b(t) = n_b + k^2t^2n_a(n_b+1), \tag{4.18}$$

$$\bar{n}_c(t) = k^2t^2n_a(n_b+1) \tag{4.19}$$

and

$$\langle \Delta n_a^2\rangle = k^2t^2n_a(n_b+1). \tag{4.20}$$

It may be of some interest to note that the fluctuations in the number of photons in either mode even when both n_b and n_c are zero correspond to vacuum fluctuations and are a purely quantum-mechanical effect.

5. Time variation of the density operator

In §§ 3 and 4, we used the Heisenberg picture and obtained the time variation of the respective operators. In order to study the statistics of parametric processes, we use in this section the Schrödinger picture and obtain the time evolution of the density operator. We shall in particular use the diagonal coherent state representation of the density operator (Sudarshan 1963, Glauber 1963, Mehta 1967)

$$\hat{\rho}(t) = \int \phi(\alpha, t)|\alpha\rangle\langle\alpha|\, d^2\alpha, \tag{5.1}$$

where $|\alpha\rangle$ is the normalized eigenstate of the annihilation operator $\hat{a}$ with complex eigenvalue α,

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle; \qquad \langle\alpha|\hat{a}^\dagger = \alpha^*\langle\alpha| \tag{5.2}$$

and

$$d^2\alpha = dx\, dy, \qquad \alpha = x + iy \qquad (x, y \text{ are real}). \tag{5.3}$$

The phase space distribution function $\phi(\alpha, t)$ is the inverse Fourier transform of the characteristic function for the normal ordering (Mehta and Sudarshan 1965, Glauber 1966) defined by the relation

$$\chi(\eta, t) = \mathrm{Tr}[\hat{\rho}(t)\exp(\eta\hat{a}^\dagger)\exp(-\eta^*\hat{a})]. \tag{5.4}$$

For, from (5.1) and (5.4) one finds that

$$\chi(\eta, t) = \int \phi(\alpha, t)\exp(\eta\alpha^* - \eta^*\alpha)\,\mathrm{d}^2\alpha, \tag{5.5}$$

and therefore

$$\phi(\alpha, t) = \frac{1}{\pi^2}\int \chi(\eta, t)\exp(\eta^*\alpha - \eta\alpha^*)\,\mathrm{d}^2\eta. \tag{5.6}$$

We have written equations (5.1)–(5.6) assuming that only one mode of oscillation is present. The formulation may readily be extended to a large number of modes. However, if we are interested in carrying out measurements only over a particular mode μ, we may then define the reduced density operator for that mode

$$\hat{\rho}_\mu = \mathrm{Tr}_\mu\hat{\rho} \tag{5.7}$$

where Tr_μ denotes trace over all modes except the mode μ. In what follows, we obtain the time evolution of the reduced density operators for the signal as well as for the pump mode. Since the interaction is symmetric in the signal and the idler modes, the density operator for the idler mode will behave in a manner identical with that for the signal mode. For this reason we shall not consider the case of the idler mode separately.

6. Diagonal coherent state representation for the signal mode

The normally ordered characteristic function for the signal mode may be written, using (5.4), in the form

$$\chi_b(\eta, t) = \mathrm{Tr}[\hat{\rho}_b(t)\exp(\eta\hat{b}_0^\dagger)\exp(-\eta^*\hat{b}_0)]. \tag{6.1}$$

To evaluate the expectation value on the right-hand side, we may use the Heisenberg picture and rewrite (6.1) in the form

$$\chi_b(\eta, t) = \mathrm{Tr}[\hat{\rho}_0\exp(\eta\hat{b}^\dagger(t))\exp(-\eta^*\hat{b}(t))], \tag{6.2}$$

where $\hat{\rho}_0$ is the initial total density operator. Using equations (3.9)–(3.11) and (4.5), we find that

$$\hat{b}(t) = [\hat{b}_0 - \mathrm{i}kt\hat{a}_0\hat{c}_0^\dagger + \tfrac{1}{2}k^2t^2\hat{b}_0(\hat{a}_0^\dagger\hat{a}_0 - \hat{c}_0^\dagger\hat{c}_0)]\,\mathrm{e}^{-\mathrm{i}\omega_b t}. \tag{6.3}$$

We evaluate the characteristic function in the case when the initial state of the system is a coherent state in all the three modes, ie, when

$$\hat{\rho}_0 = |\alpha_0, \beta_0, \gamma_0\rangle\langle\alpha_0, \beta_0, \gamma_0|. \tag{6.4}$$

We substitute (6.3) and (6.4) in (6.2). The trace evaluation is simplified if we could express the corresponding expression in a normally ordered form. If we retain terms only up to first order in kt, we readily find that

$$\chi_b(\eta, t) = \exp[\eta(\beta_0^* + \mathrm{i}kt\alpha_0^*\gamma_0)\,\mathrm{e}^{\mathrm{i}\omega_b t}]\exp[-\eta^*(\beta_0^* - \mathrm{i}kt\alpha_0\gamma_0^*)\,\mathrm{e}^{-\mathrm{i}\omega_b t}]. \tag{6.5}$$

From (5.6), we therefore obtain the following expression for the reduced diagonal coherent state representation of the signal mode:

$$\phi_b(\beta, t) = \delta^{(2)}(\beta-\bar{\beta}(t)) \equiv \delta[\mathrm{Re}(\beta-\bar{\beta}(t))]\delta[\mathrm{Im}(\beta-\bar{\beta}(t))], \tag{6.6}$$

where

$$\bar{\beta}(t) = (\beta_0 - \mathrm{i}kt\alpha_0\gamma_0^*)\,\mathrm{e}^{-\mathrm{i}\omega_b t} \tag{6.7}$$

is the expectation value of $\hat{b}(t)$ and $\delta^{(2)}$ denotes the two-dimensional Dirac delta function as indicated†.

We thus find that, correct to the first order in kt, the reduced density operator at time t for the signal mode corresponds to the coherent state with eigenvalue $\bar{\beta}(t)$ given by (6.7).

The reduced density operator correct to the second order in kt may also be evaluated. For this purpose we expand $\exp(\eta\hat{b}^\dagger(t))\exp(-\eta^*\hat{b}(t))$ in powers of kt retaining terms only up to k^2t^2 and obtain

$$\begin{aligned}
&\exp(\eta\hat{b}^\dagger(t))\exp(-\eta^*\hat{b}(t))\\
&\quad= \exp(\eta\hat{b}_0^\dagger\,\mathrm{e}^{\mathrm{i}\omega_b t})\{1+\mathrm{i}kt(\eta\,\mathrm{e}^{\mathrm{i}\omega_b t}\hat{a}_0^\dagger\hat{c}_0+\eta^*\,\mathrm{e}^{-\mathrm{i}\omega_b t}\hat{c}_0^\dagger\hat{a}_0)\\
&\qquad+\tfrac{1}{2}k^2t^2[\eta\,\mathrm{e}^{\mathrm{i}\omega_b t}\hat{b}_0^\dagger(\hat{a}_0^\dagger\hat{a}_0-\hat{c}_0^\dagger\hat{c}_0)-\eta^*\mathrm{e}^{-\mathrm{i}\omega_b t}(\hat{a}_0^\dagger\hat{a}_0-\hat{c}_0^\dagger\hat{c}_0)\hat{b}_0\\
&\qquad-\eta^2\,\mathrm{e}^{2\mathrm{i}\omega_b t}\hat{a}_0^{\dagger 2}\hat{c}_0^2-\eta^{*2}\,\mathrm{e}^{-2\mathrm{i}\omega_b t}\hat{c}_0^{\dagger 2}\hat{a}_0^2]-k^2t^2|\eta|^2\hat{a}_0^\dagger\hat{c}_0\hat{a}_0\hat{c}_0^\dagger\}\\
&\qquad\times\exp(-\eta^*\hat{b}_0\,\mathrm{e}^{-\mathrm{i}\omega_b t}).
\end{aligned} \tag{6.8}$$

The right-hand side of (6.8) is already in the normally ordered form except for the last term containing $\hat{a}_0^\dagger\hat{c}_0\hat{a}_0\hat{c}_0^\dagger$ which can be rewritten as $\hat{a}_0^\dagger\hat{c}_0^\dagger\hat{a}_0\hat{c}_0+\hat{a}_0^\dagger\hat{a}_0$. We therefore obtain after some simplification,

$$\begin{aligned}
&\exp(\eta\hat{b}^\dagger(t))\exp(-\eta^*\hat{b}(t))\\
&\quad= :\exp\{\eta[\hat{b}_0^\dagger+\mathrm{i}kt\hat{a}_0^\dagger\hat{c}_0+\tfrac{1}{2}k^2t^2\hat{b}_0^\dagger(\hat{a}_0^\dagger\hat{a}_0-\hat{c}_0^\dagger\hat{c}_0)]\,\mathrm{e}^{\mathrm{i}\omega_b t}\\
&\qquad-\eta^*[\hat{b}_0-\mathrm{i}kt\hat{a}_0\hat{c}_0^\dagger+\tfrac{1}{2}k^2t^2\hat{b}_0(\hat{a}_0^\dagger\hat{a}_0-\hat{c}_0^\dagger\hat{c}_0)]\,\mathrm{e}^{-\mathrm{i}\omega_b t}-|\eta|^2k^2t^2\hat{a}_0^\dagger\hat{a}_0\}:
\end{aligned} \tag{6.9}$$

where the colons denote, as usual, that the expression in between is to be taken in a normally ordered form, ie, all creation operators are to be put to the left of all annihilation operators. We have expressed (6.9) in an exponential form in order to obtain a simplified expression for the diagonal coherent state representation. However, it is understood that the results are valid only up to the second order in kt. From equations (6.2), (6.4) and (6.9), we therefore obtain

$$\chi_b(t) = \exp(\eta\bar{\beta}^*(t)-\eta^*\bar{\beta}(t)-|\eta|^2k^2t^2|\alpha_0|^2) \tag{6.10}$$

where $\bar{\beta}(t)$ is now given by

$$\bar{\beta}(t) = [\beta_0-\mathrm{i}kt\alpha_0\gamma_0^*+\tfrac{1}{2}k^2t^2\beta_0(|\alpha_0|^2-|\gamma_0|^2)]\,\mathrm{e}^{-\mathrm{i}\omega_b t}. \tag{6.11}$$

† If the solution (6.3) for $b(t)$ is taken to the first order in kt then an expression for $\phi_b(\beta, t)$ may be obtained without further approximation yielding

$$\phi_b(\beta, t) = \frac{1}{\pi k^2t^2|\alpha_0|^2}\exp\left(-\frac{|\beta-\bar{\beta}(t)|^2}{k^2t^2|\alpha_0|^2}\right).$$

This may appear to be different from that given by (6.6). However, various averages such as moments, etc, will be identical correct to the first order in kt, whether we use (6.6) or the above expression. In this respect they are identical.

The reduced diagonal coherent state representation obtained by using (5.6) is therefore given by†

$$\phi_b(\beta, t) = \frac{1}{\pi^2}\int \exp[\eta^*(\beta - \bar{\beta}(t)) - \eta(\beta^* - \bar{\beta}^*(t)) - |\eta|^2 k^2 t^2 |\alpha_0|^2]\, \mathrm{d}^2\eta$$

$$= \frac{1}{\pi\sigma^2(t)}\exp\left(-\frac{|\beta - \bar{\beta}(t)|^2}{\sigma^2(t)}\right) \tag{6.12}$$

where

$$\sigma^2(t) = k^2 t^2 |\alpha_0|^2. \tag{6.13}$$

$|\alpha_0|^2$ is the average number of photons initially present in the pump mode.

We thus find that correct to the second order in kt, the diagonal coherent state representation for the signal mode is a complex gaussian distribution with mean $\bar{\beta}(t)$ and variance $\sigma^2(t)$. The variance is thus large (small) if the average number of photons initially present in the pump mode is large (small). Large values of $|\alpha_0|^2$ correspond to the case of the parametric approximation. The distribution (6.12) tends to (6.6) for smaller values of kt, as it should.

Finally it may be noted that if the initial state is not a coherent state, but is given by

$$\hat{\rho}_0 = \int \phi_0(\alpha_0, \beta_0, \gamma_0)|\alpha_0, \beta_0, \gamma_0\rangle\langle\alpha_0, \beta_0, \gamma_0|\, \mathrm{d}^2\alpha_0\, \mathrm{d}^2\beta_0\, \mathrm{d}^2\gamma_0 \tag{6.14}$$

then the reduced diagonal coherent state density operator for the signal mode at time t will be given by

$$\phi_b(\beta, t) = \frac{1}{\pi}\int \frac{1}{\sigma^2(t)}\exp\left(-\frac{|\beta - \bar{\beta}(t)|^2}{\sigma^2(t)}\right)\phi_0(\alpha_0, \beta_0, \gamma_0)\, \mathrm{d}^2\alpha_0\, \mathrm{d}^2\beta_0\, \mathrm{d}^2\gamma_0. \tag{6.15}$$

7. Diagonal coherent state representation for the pump mode

Let us next consider the reduced density operator for the pump mode. The normally ordered characteristic function for the pump mode may be written in the form (cf equation (6.2))

$$\chi_a(\xi, t) = \mathrm{Tr}[\hat{\rho}_0 \exp(\xi \hat{a}^\dagger(t)) \exp(-\xi^* \hat{a}(t))], \tag{7.1}$$

where $\hat{a}(t)$ is obtained from equations (3.10)–(3.12) and (4.4),

$$\hat{a}(t) = [\hat{a}_0 - \mathrm{i}kt\hat{b}_0\hat{c}_0 - \tfrac{1}{2}k^2t^2\hat{a}_0(\hat{b}_0^\dagger\hat{b}_0 + \hat{c}_0^\dagger\hat{c}_0 + 1)]\, \mathrm{e}^{-\mathrm{i}\omega_a t}. \tag{7.2}$$

If we substitute (7.2) in the expression $\exp(\xi\hat{a}^\dagger(t))\exp(-\xi^*\hat{a}(t))$, expand in powers of kt and retain terms up to k^2t^2, we obtain, after some straightforward calculations and rearrangement of terms, the following expression:

$$\exp(\xi\hat{a}^\dagger(t))\exp(-\xi^*\hat{a}(t))$$

$$= :\exp\{\xi\, \mathrm{e}^{\mathrm{i}\omega_a t}[\hat{a}_0^\dagger + \mathrm{i}kt\hat{b}_0^\dagger\hat{c}_0^\dagger - \tfrac{1}{2}k^2t^2\hat{a}_0^\dagger(\hat{b}_0^\dagger\hat{b}_0 + \hat{c}_0^\dagger\hat{c}_0 + 1)]$$

$$-\xi^*\, \mathrm{e}^{-\mathrm{i}\omega_a t}[\hat{a}_0 - \mathrm{i}kt\hat{b}_0\hat{c}_0 - \tfrac{1}{2}k^2t^2\hat{a}_0(\hat{b}_0^\dagger\hat{b}_0 + \hat{c}_0^\dagger\hat{c}_0 + 1)]\}:. \tag{7.3}$$

† Had we used expression (6.8), $\phi_b(\beta, t)$ would have contained a Dirac delta function and its derivatives. However, again, up to second order in kt we would obtain identical results for various averages (cf footnote on p 613).

We now substitute (7.3) and (6.4) in (7.1) and find that the normally ordered characteristic function for the pump mode is given by

$$\chi_a(\xi, t) = \exp(\xi\bar{\alpha}^*(t) - \xi^*\bar{\alpha}(t)) \tag{7.4}$$

where $\bar{\alpha}(t)$ is, as before, the expectation value of $\hat{a}(t)$

$$\bar{\alpha}(t) = [\alpha_0 - ikt\beta_0\gamma_0 - \tfrac{1}{2}k^2t^2\alpha_0(|\beta_0|^2 + |\gamma_0|^2 + 1)]\,e^{-i\omega_a t}. \tag{7.5}$$

The reduced density operator for the pump mode is therefore given by

$$\phi_a(\alpha, t) = \delta^{(2)}(\alpha - \bar{\alpha}(t)). \tag{7.6}$$

It is to be noted that even to the second order in kt, the reduced density operator for the pump mode at time t, corresponds to the coherent state with eigenvalue $\bar{\alpha}(t)$. Of course if the initial state is not a coherent state (6.4), but is given by (6.14), the reduced diagonal coherent state representation for the pump mode would be given by

$$\phi_a(\alpha, t) = \int \delta^{(2)}(\alpha - \bar{\alpha}(t))\phi_0(\alpha_0, \beta_0, \gamma_0)\,d^2\alpha_0\,d^2\beta_0\,d^2\gamma_0. \tag{7.7}$$

In the above consideration we have obtained the reduced density operators for the various modes. It is also possible to obtain the time evolution of the total density operator of the system in the diagonal coherent state representation. The mathematics involved is, however, quite cumbersome. The reduced density operator for the combined signal and idler can however be readily evaluated. Assuming again that the initial state of the system is a coherent state $|\alpha_0, \beta_0, \gamma_0\rangle\langle\alpha_0, \beta_0, \gamma_0|$, we find in this case that the characteristic function for the normal ordering is given by

$$\begin{aligned}\chi_{bc}(\eta, \xi, t) &= \mathrm{Tr}[\hat{\rho}\exp(\eta\hat{b}^\dagger(t) + \xi\hat{c}^\dagger(t))\exp(-\eta^*\hat{b}(t) - \xi^*\hat{c}(t))] \\ &= \exp[\eta\bar{\beta}^*(t) + \xi\bar{\gamma}^*(t) - \eta^*\bar{\beta}(t) - \xi^*\bar{\gamma}(t) - \sigma^2(t)(|\xi|^2 + |\eta|^2)]\end{aligned} \tag{7.8}$$

where

$$\bar{\beta}(t) = [\beta_0 - ikt\alpha_0\gamma_0^* + \tfrac{1}{2}k^2t^2\beta_0(|\alpha_0|^2 - |\gamma_0|^2)]\,e^{-i\omega_b t} \tag{7.9}$$

$$\bar{\gamma}(t) = [\gamma_0 - ikt\alpha_0\beta_0^* + \tfrac{1}{2}k^2t^2\gamma_0(|\alpha_0|^2 - |\beta_0|^2)]\,e^{-i\omega_c t} \tag{7.10}$$

and

$$\sigma^2(t) = k^2t^2|\alpha_0|^2. \tag{7.11}$$

Hence the reduced diagonal coherent state representation of the combined signal and idler modes is given by

$$\phi_{bc}(\beta, \gamma, t) = \frac{1}{\pi^2\sigma^4}\exp\left(-\frac{|\beta - \bar{\beta}(t)|^2 + |\gamma - \bar{\gamma}(t)|^2}{\sigma^2(t)}\right). \tag{7.12}$$

It may be noted that $\phi_{bc}(\beta, \gamma, t) = \phi_b(\beta, t)\phi_c(\gamma, t)$ ie, that the two modes are uncoupled.

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